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Twisted h-spacetimes and invariant equations

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Abstract

We analyze the h-deformations of the Lorentz group and their associated spacetimes. We prove that they have a twisted character and give explicitly the twisting matrices. After studying the representations of one of the deformed spacetime algebras, we discuss the Klein-Gordon operator. It is found that the h-deformed d'Alembertian has plane wave solutions of the same form as the standard ones. We also give explicit expressions for the h-gamma matrices defining the associated Dirac equations.

1 Introduction

It is about fifty years since the first attempts to discretize the spacetime manifold were probably first made in [1], where the introduction of a smallest unit of length in spacetime led to non-commutative spacetime coordinates. The quantum structure of spacetime is still being discussed from different viewpoints [2] (see also [3] and references therein) which quite often turn out to be associated with various aspects of non-commutative geometry [4]. This has been used very recently to propose an action unifying gravity and the standard model at very high energies [5]. A direct introduction of a spacetime lattice structure breaks the relativistic invariance of the theory, but the recent generalization of the Lie groups to deformed ('quantum') groups gives rise to another possibility of constructing non-commutative spacetime coordinates in order to find the algebraic analogues of the special relativity relations.

The initial step of the quantum group approach consists in selecting a deformed Lorentz group and then defining a quantum Minkowski spacetime algebra by using covariance arguments similar to those used to define the 'quantum' plane [6, 7]. Although the best known and well developed deformation with dimensionless parameter [8, 9] is based on the q-deformed $SL_q(2)$ group, there are a variety of other deformations and even a classification scheme [10, 11, 12] (see also [13]). However in the

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papers mentioned above different techniques are used which do not permit an easy comparison of the results obtained.

The aim of this paper is to provide a detailed analysis of h-deformed Minkowski spacetimes and their properties, which are related to the twisted $SL_h(2)$ [12, 14]. The h-deformation is specially simple due to the natural $*$-structure [15, 14] of the h-spacetime algebras, which is absent in general. Recently, and in the framework of [10, 11], a formal solution to the deformed Klein-Gordon and Dirac equation has been proposed [15] by introducing 'non-commutative eigenvalues' of the deformed spacetime derivatives. However, the introduction of simplifying commutation relations for deformed spacetime coordinates and derivatives was done in a non-covariant way: the set of commutation relations of coordinates and derivatives with 'non-commutative eigenvalues' is not preserved under deformed Lorentz group transformation (coaction). We shall consider also this problem in our framework and find a solution consistent with covariance. The problem of the realization of Dirac matrices will also be solved naturally for the twisted deformation.

The paper is organized as follows. In Sec. 2 we recall the properties of the h-deformed Minkowski spacetime [12, 14] following the R-matrix formalism [16, 17]. The different deformed Lorentz groups and corresponding Minkowski spaces as well as their properties are directly related to a set of four R-matrices $R^{(j)}$ (j=1,2,3,4), solutions to the Yang-Baxter equation (YBE) and FRT-relations. Selecting the R-matrices associated with $SL_h(2)$ [12] we are able to determine a complete and unique set of covariant commutation relations among coordinates, derivatives and one-forms.

Sec. 3 gives the explicit expressions for the metric matrices for coordinates and derivatives, and shows how the mild (twisted) nature of the deformation is also encoded in the dilatation operator. Sec. 4 describes the h-deformed spacetime algebras and representations. Sec. 5 includes a general discussion of the Lorentz group deformations connected with triangular R-matrices and shows how the twisted nature of $SL_h(2)$ is also incorporated in the h-Lorentz groups. The properties of the resulting $\mathcal{R}$-matrix (16×16) are derived as consequences of the corresponding properties of its components, triangularity and twisting character, and the twisting matrices are constructed. Using the exchange $\mathcal{R}$-matrix and a general approach to the construction of deformed Dirac operators [17], the problem of finding the h-Dirac matrices is solved in Sec. 6. Sec. 7 is devoted to the study of the solutions of the h-deformed Klein-Gordon equation in a way consistent with covariance. Finally, the paper closes with some conclusions.

2 h-deformed Minkowski algebras and differential operators

We review first the properties of the two deformed Minkowski spaces [12, 14] associated with the 'Jordanian' or 'non-standard' h-deformation, $SL_h(2)$, of $SL(2,\mathbb{C})$ [18, 20, 19, 21]. The $GL_h(2)$ deformation is defined as the associative algebra generated by the entries a, b, c, d of a matrix M, the commutation properties of which

may be expressed by an 'FRT' equation, $R_{12}M_1M_2 = M_2M_1R_{12}$ [22], where R is the triangular solution of the Yang-Baxter equation

$$R_h = \begin{bmatrix} 1 & -h & h & h^2 \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \ , \ \hat{R}_h \equiv \mathcal{P}R_h = \begin{bmatrix} 1 & -h & h & h^2 \\ 0 & 0 & 1 & h \\ 0 & 1 & 0 & -h \\ 0 & 0 & 0 & 1 \end{bmatrix} \ , \ \mathcal{P}R_h\mathcal{P} = R_h^{-1} \ , \tag{2.1}$$

and $\mathcal{P}$ is the permutation operator ($\mathcal{P}_{ij,kl} = \delta_{il}\delta_{jk}$, $\mathcal{P} = \mathcal{P}^{-1}$, $\mathcal{P}(A\otimes B)\mathcal{P} = B\otimes A$ if the entries of A and B commute). The commutation relations of the algebra generators in M are

$$\begin{array}{lll} [a,b] = h(\xi - a^2) \quad , & [a,c] = hc^2 \quad , & [a,d] = hc(d-a) \quad , \\ [b,c] = h(ac+cd) \quad , & [b,d] = h(d^2 - \xi) \quad , & [c,d] = -hc^2 \quad ; \end{array} \tag{2.2}$$

$$\xi \equiv det_h M = ad - cb - hcd \quad . \tag{2.3}$$

Setting $\xi = 1$ reduces $GL_h(2)$ to $SL_h(2)$. The matrix $\hat{R}_h$ (being triangular, $\hat{R}_h^2 = I_4$) has two eigenvalues (1 and -1) and a spectral decomposition in terms of a rank three projector P_{h+} and a rank one projector P_{h-},

$$\hat{R}_h = P_{h+} - P_{h-} \quad , \quad P_{h\pm}\hat{R}_h = \pm P_{h\pm} \quad , \quad P_{h\pm} = \frac{1}{2}(I \pm \hat{R}_h) \quad . \tag{2.4}$$

The deformed determinant ξ in (2.3) may be then expressed as $(det_h M)P_{h-} := P_{h-}M_1M_2$. The following relations for the h-symplectic metric have an obvious equivalent in the undeformed case:

$$\epsilon_h M^t \epsilon_h^{-1} = M^{-1} \ , \ \epsilon_h = \begin{pmatrix} h & 1 \\ -1 & 0 \end{pmatrix} \ , \ \epsilon_h^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix} \ , \ P_{h-\,ij,kl} = \frac{-1}{2}\epsilon_{h\,ij}\epsilon^{-1}_{h\,kl} \, . \tag{2.5}$$

Deformed 'groups' related with different values of $h \in \mathbb{C}$ are equivalent and their R_h matrices are related by a similarity transformation. Thus, from now on, we shall take $h \in \mathbb{R}$.

The determination of a complete set of deformations of the Lorentz group [10, 12] requires replacing [11, 8, 9] the $SL(2,\mathbb{C})$ matrices A in the undeformed expression $K' = AKA^\dagger$ ($K = K^\dagger = \sigma_\mu x^\mu$) by the generator matrix M of a deformation of $SL(2,\mathbb{C})$, and the characterization of all possible commutation relations among the generators (a,b,c,d) of M and (a^*,b^*,c^*,d^*) of $M^\dagger$. In particular, for the deformed Lorentz groups associated with $SL_h(2)$, the R-matrix form of these commutation relations may be expressed by

$$\begin{array}{ll} R_h M_1 M_2 = M_2 M_1 R_h \quad , & M_1^\dagger R^{(2)} M_2 = M_2 R^{(2)} M_1^\dagger \quad , \\ M_2^\dagger R^{(3)} M_1 = M_1 R^{(3)} M_2^\dagger \quad , & R_h^\dagger M_1^\dagger M_2^\dagger = M_2^\dagger M_1^\dagger R_h^\dagger \quad , \end{array} \tag{2.6}$$

where $R^{(3)\,\dagger} = R^{(2)} = \mathcal{P}R^{(3)}\mathcal{P}$ ('reality' condition for $R^{(3)}$). The consistency of these relations is assured if $R^{(3)}$ satisfies the FRT equation (see [17, 23] in this respect)

$$R_{h\,12}R^{(3)}_{13}R^{(3)}_{23} = R^{(3)}_{23}R^{(3)}_{13}R_{h\,12} \quad . \tag{2.7}$$

This equation, considered as an FRT equation, indicates that $R^{(3)}$ is a representation of $GL_h(2)$, $(M_{ij})_{\alpha\beta} = R^{(3)}_{i\alpha,j\beta}$. The solutions of these equations[2] [12, 10] ($h, r \in \mathbb{R}$),

$$\mathbf{1.} \quad R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{2.} \quad R^{(3)} = \begin{bmatrix} 1 & 0 & -h & 0 \\ -h & 1 & 0 & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h & 1 \end{bmatrix} \quad , \tag{2.8}$$

characterize the two h-deformed Lorentz groups, which will be denoted $L_h^{(1)}$ and $L_h^{(2)}$ respectively.

To introduce the deformed Minkowski *algebra* $\mathcal{M}_h^{(j)}$ associated with the h-Lorentz group $L_h^{(j)}$ (j=1,2) we extend $K' = AKA^\dagger$ above by stating that in the deformed case the corresponding $K^{(j)}$ generates a comodule algebra for the coaction ϕ defined by

$$\phi : K \longmapsto K' = MKM^\dagger \ , \ K'_{is} = M_{ij} M^\dagger_{ls} K_{jl} \ , \ K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = K^\dagger \ , \ \Lambda = M \otimes M^* \ , \tag{2.9}$$

where Λ is the $L_h^{(j)}$ matrix and, as usual, it is assumed that the elements of K, which now do not commute among themselves, commute with those of M and $M^\dagger$. We now demand that the commuting properties of the entries of K are preserved by (2.9). The use of covariance arguments to characterize the algebra generated by the elements of K has been extensively used, and the resulting equations are associated with the name of reflection equations (see [24, 25] and references therein) or, in a more general setting, braided algebras [26, 27]. In the present $SL_h(2)$ case, the commutation properties of the entries of the hermitian matrix K generating a deformed Minkowski algebra $\mathcal{M}_h$ are given by

$$R_h K_1 R^{(2)} K_2 = K_2 R^{(3)} K_1 R_h^\dagger \quad , \tag{2.10}$$

where the $R^{(3)} = R^{(2)\,\dagger}$ matrices are those given in (2.8); using (2.6), it is easy to check that (2.10) is preserved under ϕ in (2.9).

The deformed Minkowski length and metric, invariant under a Lorentz transformation (2.9) of $L_h^{(j)}$, is defined [12] through the h-deformed determinant of K

$$(det_h K) P_{h-} P_{h-}^\dagger = -P_{h-} K_1 \hat{R}^{(3)} K_1 P_{h-}^\dagger \quad , \tag{2.11}$$

where $P_{h-} P_{h-}^\dagger$ is a projector since $(P_{h-} P_{h-}^\dagger)^2 = \left(\frac{2+h^2}{2}\right)^2 P_{h-} P_{h-}^\dagger$. $det_h K$ is invariant, central and, since $\hat{R}^{(3)}$ and K are hermitian, real; thus, it defines the *deformed Minkowski length* l_h for the h-deformed spacetimes $\mathcal{M}_h^{(j)}$.

To describe the differential calculus on h-Minkowski spaces, we need expressing the different commutation relations among the fundamental objects: deformed coordinates, derivatives and one-forms. Following the approach of [16, 17, 28] to the differential calculus on Minkowski algebras associated with the standard deformation

[2] The first sentence in [12] in the Remark after (36) should be deleted.

$SL_q(2)$, we introduce the reflection equations expressing the commutation relations defining the algebras of h-derivatives and h-one-forms (h-differential calculi have been considered in [21] and in [15] for quantum N-dimensional homogeneous spaces).

The derivatives are expressed in terms of an object Y transforming covariantly (cf. (2.9)) *i.e.*,

$$Y \longmapsto Y' = (M^\dagger)^{-1} Y M^{-1} \quad , \quad Y = \begin{bmatrix} \partial_\alpha & \partial_\gamma \\ \partial_\beta & \partial_\delta \end{bmatrix} \quad ; \tag{2.12}$$

Their commutation properties are described by

$$R_h^\dagger Y_2 R^{(2)\,-1} Y_1 = Y_1 R^{(3)\,-1} Y_2 R_h \quad , \tag{2.13}$$

where $R^{(3)} = \mathcal{P} R^{(2)} \mathcal{P}$ is given in (2.8), which is preserved under the h-Lorentz coaction; they are explicitly given in [14].

The commutation relations among the entries of K and Y may be expressed by an inhomogeneous reflection equation [16, 17]

$$Y_2 R_h K_1 R^{(2)} = R^{(3)} K_1 R_h^\dagger Y_2 + R^{(3)} \mathcal{P} \quad , \tag{2.14}$$

which extends to the h-deformed case the undeformed relation $\partial_\mu x^\nu = \delta_\mu^\nu + x^\nu \partial_\mu$. Eq. (2.14) is consistent with the commutation relations defining the algebras $\mathcal{M}_h^{(j)}$, $\mathcal{D}_h^{(j)}$, and is invariant under h-Lorentz transformations. This is seen by multiplying eq. (2.14) by $(M_2^\dagger)^{-1} M_1$ from the left and by $M_1^\dagger M_2^{-1}$ from the right and using the commutation relations in (2.6). Eq. (2.14) is unique [12] due to the triangularity of R_h (in contrast with the situation for q-deformation, where covariance allows for more than one type of these equations [17, 28]). In contrast with the q-deformation [9, 17] (see also [29]), it is a common feature of all h-deformed Minkowski spaces that the transformation properties for ‘coordinates’ and ‘derivatives’ are consistent with their simultaneous hermiticity [14, 15].

To determine the commutation relations for the h-de Rham complex we now introduce the exterior derivative d [9, 17, 16]. The algebra of h-forms is generated by the entries of a matrix dK. Since d commutes with the Lorentz coaction

$$dK' = M \, dK \, M^\dagger \quad . \tag{2.15}$$

Applying d to eq. (2.10) we obtain

$$R_h dK_1 R^{(2)} K_2 + R_h K_1 R^{(2)} dK_2 = dK_2 R^{(3)} K_1 R_h^\dagger + K_2 R^{(3)} dK_1 R_h^\dagger \quad . \tag{2.16}$$

Its only solution is given by

$$R_h dK_1 R^{(2)} K_2 = K_2 R^{(3)} dK_1 R_h^\dagger \tag{2.17}$$

(which implies $R_h K_1 R^{(2)} dK_2 = dK_2 R^{(3)} K_1 R_h^\dagger$). From eq. (2.17) and d^2=0, it follows that

$$R_h dK_1 R^{(2)} dK_2 = -dK_2 R^{(3)} dK_1 R_h^\dagger \quad . \tag{2.18}$$

Again, it is easy to check that these relations are invariant under hermitian conjugation. Notice that eqs. (2.10), (2.17) and (2.18) have the same R-matrix structure. In the h-deformed case, the equation giving the commutation relations among the generators of two differential algebras is determined only by the transformation (covariant or contravariant) law of these generators. Thus, as a consequence of the triangularity of $SL_h(2)$, there are [12, 14] only three types of reflection equations, those in (2.10), (2.13) and (2.14). In contrast, this number is larger for the $SL_q(2)$ based q-deformation.

For the braiding properties of these algebras we refer to [14].

3 Differential operators and the dilatation operator as a measure of the deformation

It is possible to construct from the previous differential algebras some invariant operators by using the $L_h^{(j)}$-invariant scalar product of *contravariant* (transforming as the matrix K, eq. (2.9)) and *covariant* ($Y \mapsto Y' = (M^\dagger)^{-1} Y M^{-1}$) matrices (Minkowski four-vectors) which may be defined as the quantum trace [22, 30] of a matrix product [17]. The h-deformed trace [12] of a matrix B is given by

$$tr_h(B) := tr(D_h B) \quad , \quad D_h := tr_{(2)}(\mathcal{P}((R_h^{t_1})^{-1})^{t_1}) = \begin{pmatrix} 1 & -2h \\ 0 & 1 \end{pmatrix} \ , \tag{3.1}$$

where $tr_{(2)}$ means trace in the second space. tr_h is invariant under the quantum group coaction $B \mapsto MBM^{-1}$ since the expression of D_h above guarantees that $D_h^t = M^t D_h^t (M^{-1})^t$ is fulfilled. This is not the only possible definition; for an object C transforming as $C \mapsto (M^\dagger)^{-1} C M^\dagger$ we might define another invariant trace $\tilde{tr}_h$ by

$$\tilde{tr}_h C = tr \tilde{D}_h C \quad \text{with} \quad \tilde{D}_h = \begin{pmatrix} 1 & 0 \\ -2h & 1 \end{pmatrix} = D_h^\dagger \quad . \tag{3.2}$$

The invariant h-determinant is related with the h-trace [12]. Consider

$$K^\epsilon_{ij} := \hat{R}^\epsilon_{h\,ij,kl} K_{kl} \quad , \quad \hat{R}^\epsilon_h := (1 \otimes (\epsilon_h^{-1})^t) \hat{R}^{(3)} (1 \otimes (\epsilon_h^{-1})^\dagger) \quad , \quad \hat{R}^\epsilon_{12} = \hat{R}^\epsilon_{21} \tag{3.3}$$

(the explicit expressions of $\hat{R}_h^{\epsilon\,-1}$ for (2.8) are given in eqs. (6.19), (6.21)). Then, if K is contravariant [eq. (2.9)], K^ϵ is covariant *i.e.*, $K^\epsilon \mapsto (M^\dagger)^{-1} K^\epsilon M^{-1}$. This may be checked by using the property of $\hat{R}^\epsilon_h$,

$$\hat{R}^\epsilon_h (M \otimes (M^\dagger)^t) = ((M^\dagger)^{-1} \otimes (M^{-1})^t) \hat{R}^\epsilon_h \ \text{ or } \ \hat{R}^\epsilon_h M_1 M_2^* = (M_1^\dagger)^{-1} (M_2^{-1})^t \hat{R}^\epsilon_h \ , \tag{3.4}$$

which follows from the preservation of the h-symplectic metric ϵ_h. Now, using the expression of ϵ_h in (2.5), $(P_{h-})_{ij,kl} = -\frac{1}{2} \epsilon_{h\,ij} \epsilon^{-1}_{h\,kl}$ and $D_h = -\epsilon_h (\epsilon_h^{-1})^t$, it follows that the *$h$-deformed Minkowski length* l_h and *h-metric* g_h are given by

$$l_h := det_h K = \frac{1}{2+h^2} tr_h K K^\epsilon \equiv g_{h\,ij,kl} K_{ij} K_{kl} \quad , \quad g_{h\,ij,kl} = \frac{1}{2+h^2} D_{h\,si} \hat{R}^\epsilon_{h\,js,kl} \ . \tag{3.5}$$

The h-metric is preserved under h-Lorentz transformations $\Lambda = M \otimes M^*$.

Similarly, the *h-deformed d'Alembertian* may be introduced by

$$\Box_h := det_h Y = \frac{1}{2+h^2} tr_h(Y^\epsilon Y) \quad , \quad Y^\epsilon = (\hat{R}^\epsilon_h)^{-1} Y \quad , \quad Y^\epsilon \longmapsto M Y^\epsilon M^\dagger \quad . \tag{3.6}$$

As l_h, $\Box_h$ is h-Lorentz invariant, real and central in the algebra $\mathcal{D}_h^{(j)}$ of derivatives. This definition of $\Box_h$ leads to the Minkowski metric g^Y for the derivatives

$$\Box_h = g^Y_{h\,ik,mn} Y_{mn} Y_{ik} \quad , \quad g^Y_{h\,ik,mn} = \frac{1}{2+h^2} D_{h\,kj} \hat{R}^{\epsilon\,-1}_{h\,ji,mn} \; . \tag{3.7}$$

Using the property $tr_h Y^\epsilon Y = \tilde{tr}_h Y Y^\epsilon$, one gets another expression for g^Y which will be used below

$$g^Y_{h\,ik,mn} = \frac{1}{2+h^2} D_{h\,mj} \hat{R}^{\epsilon\,-1}_{h\,nj,ik} \; . \tag{3.8}$$

It is easy to check from eqs. (3.7) and (3.8) that g^Y satisfies $(g^Y)^t = \mathcal{P} g^Y \mathcal{P}$. Using eqs. (6.19), (6.21), the $g^{Y\,(j)}$ explicitly read

$$\mathbf{1.}\; g^{Y\,(1)} = \tfrac{1}{2+h^2} \begin{bmatrix} r-h^2 & -h & h & 1 \\ h & 0 & -1 & 0 \\ -h & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \; ; \quad \mathbf{2.}\; g^{Y\,(2)} = \tfrac{1}{2+h^2} \begin{bmatrix} -5h^2 & 0 & 2h & 1 \\ 2h & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \; . \tag{3.9}$$

Other invariant differential operators given by the h-trace are the *exterior derivative* d,

$$d = tr_h(dK\,Y) = d\alpha\partial_\alpha + d\beta\partial_\beta + d\gamma\partial_\gamma + d\delta\partial_\delta - 2h(d\gamma\partial_\alpha + d\delta\partial_\beta) \quad , \tag{3.10}$$

and the *dilatation operator* s,

$$s = tr_h(K\,Y) = \alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma + \delta\partial_\delta - 2h(\gamma\partial_\alpha + \delta\partial_\beta) \quad . \tag{3.11}$$

The dilatation operator has a special significance since it may be considered as a measure of the 'strength' of the deformation. In the undeformed case $s = x^\mu \partial_\mu$ and accordingly $s x^\nu = x^\nu(1+s)$. For the $SL_q(2)$-based deformations of the Minkowski space, the commutation relations of s_q with the generators of $\mathcal{M}_q$ algebra acquire extra q-dependent terms [17] which do not appear in the above undeformed case. For the case of the twisted Minkowski space of [31], the commutation relations of s and K turn out to be [17] as in the undeformed case. We shall now prove that this behaviour is in fact a general property of the deformed Minkowski algebras which are defined through a triangular R-matrix (this is our case here since $SL_h(2)$ is triangular [18, 19]).

Proposition 3.1

Let $\mathcal{M}$ be a deformation of the Minkowski space based on a triangular R-matrix, and s the dilatation operator defined by (3.11). Then,

$$sK = K(s+1) \quad , \quad e^{\alpha s} K e^{-\alpha s} = e^{\alpha} K \ , \ \alpha \in \mathbb{R} \ . \tag{3.12}$$

Proof :
The invariance of the h-trace implies

$$tr_h B = tr_h M B M^{-1} = tr_{h(1)} R_h B R_h^{-1} = tr_{h(1)} R^{(3)} B R^{(3)\,-1} \tag{3.13}$$

since R_h and $R^{(3)}$ are representations of $GL_h(2)$. Multiplying eq. (2.14) by $R_{21}K_2$ from the left and by $R^{(2)\,-1}$ from the right we get

$$R_{21}K_2Y_2R_{12}K_1 = R_{21}K_2R^{(3)}K_1R_{12}^{\dagger}Y_2R^{(2)\,-1} + \mathcal{P}R_{12}K_1 \quad . \tag{3.14}$$

We now use eq. (2.10) in the first term of the $r.h.s.$ and take the h-trace in the second space. Using that for a triangular R-matrix $R_{12} = R_{21}^{-1}$ the $tr_{h(2)}$ satisfies $tr_{h(2)} R_{21} B R_{12} = tr_{h(2)} B$, we obtain

$$s\,K_1 = K_1\, tr_{h(2)} R^{(2)} K_2 Y_2 R^{(2)\,-1} + K_1\, tr_{h(2)} \mathcal{P} R_{12} \quad . \tag{3.15}$$

Since $tr_{h(2)}\mathcal{P}R_{12} \equiv tr_{(2)}(I_2 \otimes D_h)\hat{R}_h = I_2$ (for R_q this is $q^2 I_2$), we obtain $sK = K(1+s)$, *q.e.d.*.

4 h-Minkowski algebras and representations

Using the $R^{(3)}$ matrices given in (2.8) in eq. (2.10), the Minkowski algebras associated with $SL_h(2)$ explicitly read

1. $\mathcal{M}_h^{(1)}$:

$$\begin{array}{ll} [\alpha,\beta] = -h\beta^2 - r\beta\delta + h\delta\alpha - h\beta\gamma + h^2\delta\gamma\ , & [\alpha,\delta] = h(\delta\gamma - \beta\delta)\ , \\ [\alpha,\gamma] = h\gamma^2 + r\delta\gamma - h\alpha\delta + h\beta\gamma - h^2\beta\delta\ , & [\beta,\delta] = h\delta^2\ , \\ [\beta,\gamma] = h\delta(\gamma+\beta) + r\delta^2\ , & [\gamma,\delta] = -h\delta^2\ ; \end{array} \tag{4.1}$$

2. $\mathcal{M}_h^{(2)}$:

$$\begin{array}{ll} [\alpha,\beta] = 2h\alpha\delta + h^2\beta\delta \quad , & [\alpha,\delta] = 2h(\delta\gamma - \beta\delta) \quad , \\ [\alpha,\gamma] = -h^2\delta\gamma - 2h\delta\alpha \quad , & [\beta,\delta] = 2h\delta^2 \quad , \\ [\beta,\gamma] = 3h^2\delta^2 \quad , & [\gamma,\delta] = -2h\delta^2 \quad . \end{array} \tag{4.2}$$

In each case there exists a subalgebra generated by β, γ, δ which will be denoted by $\mathcal{A}^{(j)}$ (j=1,2).

The h-deformed Minkowski length determines quadratic central elements for both $\mathcal{M}_h^{(1)}$ and $\mathcal{M}_h^{(2)}$,

$$l_h^{(1)} = \frac{2}{h^2+2}(\alpha\delta - \beta\gamma + h\beta\delta) \quad , \tag{4.3}$$

$$l_h^{(2)} = \frac{2}{h^2+2}(\alpha\delta - \beta\gamma + 2h\beta\delta) \quad . \tag{4.4}$$

For $\mathcal{M}_h^{(2)}$, it is possible to find a linear central real element given by

$$\zeta := \beta + \gamma - \frac{3}{2}h\delta \quad . \tag{4.5}$$

We may extract from $\mathcal{A}^{(2)}$ the following set of hermitian generators

$$x := h\delta \quad , \quad y := -i(\beta - \gamma) \quad , \quad \zeta := \beta + \gamma - \frac{3}{2}h\delta \quad , \tag{4.6}$$

($\alpha^* = \alpha$, $\delta^* = \delta$, $\beta^* = \gamma$, eq. (2.9)) and hence

$$\delta = x/h \quad , \quad \beta = \frac{1}{2}(\frac{3}{2}x + iy + \zeta) \quad , \quad \gamma = \frac{1}{2}(\frac{3}{2}x - iy + \zeta) \; . \tag{4.7}$$

This makes easier to discuss the representations of $\mathcal{M}_h^{(2)}$. Since ζ is central, we can represent it by a real number $\zeta \in \mathbb{R}$. The commutation relations of x, x^{-1} and y are

$$[x, y] = 4ix^2 \quad , \quad [y, x^{-1}] = 4i \quad . \tag{4.8}$$

The second commutator is of the canonical Heisenberg type, so the unique (up to unitary equivalence) irreducible representation is $L_2(\mathbb{R})$ (Heisenberg algebra) with x as the multiplication operator and

$$y = -4ix\frac{d}{dx}x = -4i(x^2\frac{d}{dx} + x) \quad . \tag{4.9}$$

We may now extend this representation of $\mathcal{A}^{(2)}$ to the whole algebra $\mathcal{M}_h^{(2)}$. The generator α in this irreducible representation is defined by fixing the quadratic central element $l_h^{(2)} \in \mathbb{R}$ (if we consider $\mathcal{M}_h$ as a h-deformed momenta algebra this condition corresponds to fixing the mass-shell). Using (4.6) in the expression of $l_h^{(2)}$ [eq. (4.4)] one finds

$$x\alpha = h(\tfrac{2+h^2}{2})l_h^{(2)} + h(\beta - 2x)\gamma \tag{4.10}$$

and, using (4.7) to express β and γ in terms of x, y and ζ

$$\alpha = \frac{h}{4}(\frac{1}{x}(y^2 + 2i\{y, x\}_+) - \frac{23}{4}x - \zeta + \frac{1}{x}(\zeta^2 + (4+h^2)l_h^{(2)})) \quad . \tag{4.11}$$

There is also a one-dimensional representation with

$$\alpha \in \mathbb{R} \quad , \quad \delta = 0 \quad , \quad \beta = \gamma^* \in \mathbb{C} \; . \tag{4.12}$$

These representations are very different from those of q-deformed Minkowski spaces [17], but also from other twisted Minkowski spaces as that of [31] (called $\mathcal{M}_p$ in [17]).

5 Properties of the h-Lorentz algebras: triangularity and twisted character

In the earliest papers [8, 9] on the deformed spacetime algebra $\mathcal{M}_q$ (associated with $SL_q(2)$) the commutation relation of the quantum coordinates (generators of $\mathcal{M}_q$) were written in an exchange algebra form

$$\mathcal{R}_{12} X_1 X_2 = \mu X_2 X_1 \tag{5.1}$$

in terms of a 16×16 R-matrix $\mathcal{R}$ [9] (or the corresponding projector [8]) and a four component column vector X. The convenience of the reflection equation formalism (see *e.g.* [25]) to achieve an unified description of the different deformations of the Lorentz group and the Minkowski space for dimensionless deformation parameters was exhibited in a series of papers [16, 17, 12, 32]. The commutation relations of the deformed coordinates were written in the generic form

$$R^{(1)} K_1 R^{(2)} K_2 = K_2 R^{(3)} K_1 R^{(4)} \quad , \tag{5.2}$$

where the 4×4 matrices $R^{(i)}$ (i=1,2,3,4) satisfy a number of consistency conditions (as (2.7)) [23, 17, 12], and K is a 2×2 matrix representing the deformed coordinates (eq. (2.10) is a particular case of this procedure).

In order to have an explicit comparison of this approach with the above mentioned papers [8, 9] as well as to study the triangularity of $L_h^{(j)}$, it is convenient to transform the reflection equation (5.2) into the exchange algebra (5.1) and to relate the properties of the R-matrix $\mathcal{R}$ with those of the $R^{(i)}$. Provided the existence of the corresponding inverse matrices, eq. (5.2)

$$R^{(1)}_{ij,ab} K_{ac} R^{(2)}_{cb,kd} K_{dl} = K_{ja'} R^{(3)}_{ia',b'c'} K_{b'd'} R^{(4)}_{d'c',kl} \tag{5.3}$$

(summation over repeated indices understood) may be written in the exchange algebra form

$$K_{ac} K_{dl} = \mathcal{R}_{(dl)(ac),(ja')(b'd')} K_{ja'} K_{b'd'} \quad , \tag{5.4}$$

where

$$\mathcal{R}_{(dl)(ac),(ja')(b'd')} = (R^{(1)})^{-1}_{ab,ij} (R^{(2)\,t_2})^{-1}_{kb,cd} R^{(3)}_{ia',b'c'} R^{(4)}_{d'c',kl} \tag{5.5}$$

(see [33] for an analogous discussion in the particular case of the q-Minkowski space of [8, 9]). Eq. (5.4) corresponds to the form (5.1) once the matrix elements K_{ab} are understood as components of a column vector labelled by a pair of indices $[(ab) = (11), (12), (21), (22)]$. Due to the various consistency conditions [17, 23] that the $R^{(i)}$ matrices have to satisfy, the $\mathcal{R}$ satisfies the Yang-Baxter equation.

Remark. In the case where all the $R^{(j)}$ (j=1,2,3,4) matrices in (5.2) are expressed in terms of the standard R_q matrix [22] for $GL_q(2)$, and which leads to the Minkowski algebra $\mathcal{M}_q$ [8, 9], considerations on the projectors [9] lead to the introduction of two different 16×16 R-matrices associated with the q-Lorentz group. In this case, the matrix $\mathcal{R}$ of (5.5) coincides with one of them (after a change of basis it corresponds

to R_{II} in [9]). To find the other R-matrix (which is the one used for the definition of the q-Lorentz group in [8]) the starting point is not (5.2), but the reflection equation which provides the commutation relations between the differentials dK ([16], eq. (37)). Nevertheless, the procedure leading to (5.5) is the same (for a comparison of the eqs. in [9] with this formalism see [28]).

The triangularity property of the R-matrix, $R_{12}R_{21} = I$ [34], is important in the classification scheme of deformed Poincaré algebras [35] and has also been used in a recent proposal to solve the Klein-Gordon and Dirac equations in deformed Minkowski spaces [15]. The description of $\mathcal{M}_h^{(j)}$ of Sec.2 related to the 'triangular' quantum group $SL_h(2)$ is, on the other hand, based on the matrices $R^{(j)}$, of which $R^{(1)} = R_h$ and $R^{(4)}$ are triangular. It is then natural to ask whether the triangularity of the 16×16 matrix $\mathcal{R}$ is implied by the triangularity of the 4×4 R-matrices. Due to the general consistency relation $R^{(3)} = \mathcal{P}R^{(2)}\mathcal{P}$ the following proposition holds:

Proposition 5.1

Let $R^{(1)}$ and $R^{(4)}$ be triangular R-matrices ($R_{12}R_{21} = I$) and let $R^{(3)}$ and $R^{(2)}$ related by $R^{(3)} = \mathcal{P}R^{(2)}\mathcal{P}$ and satisfy the consistency conditions (2.7) (eq. (2.7) is not necessary below, but it is implicit in the introduction of $\mathcal{R}$). Then, the R-matrix $\mathcal{R}$ in (5.5) is also triangular *i.e.*,

$$\mathcal{R}_{(dl)(ac),(ja')(b'd')}\mathcal{R}_{(b'd')(ja'),(mn)(rs)} = \delta_{dr}\delta_{ls}\delta_{am}\delta_{cn} = I_{(dl)(ac),(rs)(mn)} \tag{5.6}$$

since $(\mathcal{R}_{21})_{(ac)(dl),(b'd')(ja')} = (\mathcal{R}_{12})_{(dl)(ac),(ja')(b'd')}$.

Proof:

Using (5.5), the $\mathcal{R}$ matrices in the *l.h.s.* of (5.6) are expressed in terms of $R^{(i)}$ (i=1,2,3,4) by

$$(R^{(1)})^{-1}_{ab,ij}(R^{(2)\,t_2})^{-1}_{kb,cd}R^{(3)}_{ia',b'c'}R^{(4)}_{d'c',kl}(R^{(1)})^{-1}_{jp,tm}(R^{(2)\,t_2})^{-1}_{zp,a'b'}R^{(3)}_{tn,rf}R^{(4)}_{sf,zd'}\ . \tag{5.7}$$

Thus, the proof of (5.6) reduces to some index contractions between pairs of R-matrices. We start by

$$R^{(3)}_{ia',b'c'}(R^{(2)\,t_2})^{-1}_{zp,a'b'} = R^{(2)\,t_2}_{a'b',c'i}(R^{(2)\,t_2})^{-1}_{zp,a'b'} = \delta_{ip}\delta_{zc'}\ . \tag{5.8}$$

Then we have independent contractions in which we must use the triangularity of $R^{(1)}$ and $R^{(4)}$

$$(R^{(1)})^{-1}_{ab,ij}(R^{(1)})^{-1}_{ji,tm} = (R^{(1)})^{-1}_{ab,ij}R^{(1)}_{ij,mt} = \delta_{am}\delta_{bt} \quad , \tag{5.9}$$

$$R^{(4)}_{d'c',kl}R^{(4)}_{sf,c'd'} = R^{(4)}_{d'c',kl}(R^{(4)})^{-1}_{fs,d'c'} = \delta_{kf}\delta_{ls} \quad . \tag{5.10}$$

Finally, the last contraction is similar to the first one

$$(R^{(2)\,t_2})^{-1}_{kb,cd}R^{(3)}_{bn,rk} = (R^{(2)\,t_2})^{-1}_{kb,cd}R^{(2)\,t_2}_{nr,kb} = \delta_{cn}\delta_{dr}\ . \tag{5.11}$$

q.e.d.

Remark. We stress that the proof depends only on the triangularity of $R^{(1)}$ and $R^{(4)}$ (which, in a more general setting than the present one in which we are concerned with deformed spacetimes, might even have different dimensions) and not $R^{(2)}$ ($R^{(3)} = \mathcal{P}R^{(2)}\mathcal{P}$).

Using the expression of $R_h = R^{(1)} = R^{(4)\,t}$ (eq. (2.1)) and $R^{(3)\,(j)}$ (j=1,2) (eq. (2.8)) in (5.5) the explicit form of $\mathcal{R}$ is found. The two matrices $\mathcal{R}^{(j)}$ for the h-deformations $L_h^{(j)}$ are given in the appendix, where their triangularity may be checked by direct computation.

It is well known [34, 36] that a special class of triangular R-matrices is obtained by the twisting procedure. A twisted R-matrix is obtained from a non-singular matrix F (with some extra properties) by

$$R_{12} := F\mathcal{P}F^{-1}\mathcal{P} = F\tilde{F}^{-1} \quad , \quad \tilde{F} := \mathcal{P}F\mathcal{P} \quad . \tag{5.12}$$

This is the case of the R-matrix of $SL_h(2)$ [18]. The R-matrix R_h (2.1) may be expressed in terms of a matrix F which is given in the fundamental representation by [19]

$$F = \begin{bmatrix} 1 & h/2 & -h/2 & 0 \\ 0 & 1 & 0 & h/2 \\ 0 & 0 & 1 & -h/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I + \frac{h}{2}(H \otimes E - E \otimes H) \tag{5.13}$$

where $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Since for the h-Minkowski algebras $R^{(4)} = R_h^t$ (see (2.10) and (5.2)) it follows that $R^{(4)}$ is also obtained by twisting from a matrix G related to F, $R^{(4)} = R^{(1)\,t} = \tilde{F}^{-1\,t}F^t = G\tilde{G}^{-1}$ so that

$$G = (\mathcal{P}F^t\mathcal{P})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ h/2 & 1 & 0 & 0 \\ -h/2 & 0 & 1 & 0 \\ h^2/2 & h/2 & -h/2 & 1 \end{bmatrix} \quad . \tag{5.14}$$

Let us now prove that, much in the same way as the triangularity property (Proposition 5.1), the property that $R^{(1)}$ and $R^{(4)}$ are obtained by twisting is also inherited by the 16×16 matrix $\mathcal{R}$ of the deformed Lorentz group. This property, as the previous triangularity one, is general when the $R^{(1)}$ and $R^{(4)}$ are those appearing in a reflection equation (cf. (5.2)) and $\mathcal{R}$ is the matrix appearing in the associated exchange algebra.

Proposition 5.2

Let $R^{(1)}$ and $R^{(4)}$ in (5.2) be obtained by a twisting procedure, *i.e.*, there exist two invertible matrices F and G such that

$$R^{(1)} = F\tilde{F}^{-1} \quad , \quad R^{(4)} = G\tilde{G}^{-1} \quad . \tag{5.15}$$

Then the exchange algebra 16×16 matrix $\mathcal{R}$ given by (5.5) has the same structure

$$\mathcal{R} = \mathcal{F}\tilde{\mathcal{F}}^{-1} \tag{5.16}$$

where

$$\mathcal{F}_{(dl)(ac),(rm)(sn)} = F_{ba,sr}(R^{(2)\,t_2})^{-1}_{kb,cd}\tilde{G}^{-1}_{mn,kl} \quad . \tag{5.17}$$

Proof :
Using (5.15) in (5.5) we obtain an expression of $\mathcal{R}$ as a product of two matrices

$$\begin{aligned}\mathcal{R}_{(dl)(ac),(ja')(b'd')} &= \tilde{F}_{ab,rs}F^{-1}_{rs,ij}(R^{(2)\,t_2})^{-1}_{kb,cd}R^{(3)}_{ia',b'c'}G_{d'c',mn}\tilde{G}^{-1}_{mn,kl}\\ &= [\tilde{F}_{ab,rs}(R^{(2)\,t_2})^{-1}_{kb,cd}\tilde{G}^{-1}_{mn,kl}][F^{-1}_{rs,ij}R^{(2)\,t_2}_{a'b',c'i}G_{d'c',mn}]\\ &:= \mathcal{F}_{(dl)(ac),(rm)(sn)}\mathcal{H}_{(rm)(sn),(ja')(b'd')} \quad .\end{aligned} \tag{5.18}$$

To prove the twisted character of $\mathcal{R}$ we must now check that $\mathcal{H} = \tilde{\mathcal{F}}^{-1}$. Multiplying $\tilde{\mathcal{F}}$ (obtained from (5.17), $\tilde{\mathcal{F}}_{(ac)(dl),(sn)(rm)} = \mathcal{F}_{(dl)(ac),(rm)(sn)}$) and $\mathcal{H}$, one gets

$$\begin{aligned}&\tilde{\mathcal{F}}_{(dl)(ac),(rm)(sn)}\mathcal{H}_{(rm)(sn),(ja')(b'd')} = \tilde{F}_{bd,sr}(R^{(2)\,t_2})^{-1}_{kb,la}\tilde{G}^{-1}_{nm,kc}F^{-1}_{rs,ij}R^{(2)\,t_2}_{a'b',c'i}G_{d'c',mn}\\ &\quad = \delta_{bi}\delta_{dj}\delta_{c'k}\delta_{d'c}(R^{(2)\,t_2})^{-1}_{kb,la}R^{(2)\,t_2}_{a'b',c'i} = \delta_{dj}\delta_{d'c}(R^{(2)\,t_2})^{-1}_{c'i,la}R^{(2)\,t_2}_{a'b',c'i}\\ &\quad = \delta_{dj}\delta_{d'c}\delta_{a'l}\delta_{b'a} = I_{(dl)(ac),(ja')(b'd')} \quad ,\end{aligned} \tag{5.19}$$

q.e.d.

Corollary 5.1 The h-deformations of the Lorentz group associated with $SL_h(2)$ [10, 12] are a twisting of the standard Lorentz group.

6 Dirac γ-matrices for h-Minkowski spaces

Any discussion of deformed relativistic equations requires the expression of the deformed d'Alembertian for the Klein-Gordon equation and the deformed Dirac matrices for the Dirac one. In the present case, $\Box_h$ is given by eq. (3.6). We shall devote this section to find the explicit form of the h-Dirac matrices and to prove that they satisfy the appropriate h-anticommutation properties.

The generators of a deformed Minkowski algebra $\mathcal{M}_h^{(i)}$ and the generators of the corresponding derivatives algebra $\mathcal{D}_h^{(i)}$ can be arranged in the elements of 2×2 matrices K and Y of (2.10) and (2.13). Using the set of 2×2 matrices e_{ij} (i,j=1,2) defined by $(e_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$, as a basis of the 2×2 matrices, we may write

$$Y = e_{ij}\partial_{ij} \quad . \tag{6.1}$$

Any linear transformation of the generators $\partial_{ij} \to \partial'_{mn} = A_{mn,ij}\partial_{ij}$ results in the inverse transformation for the matrices e_{ij}.

The quantum group coaction for a covariant vector $Y \to (M^\dagger)^{-1}YM^{-1}$ (with $det_hY \neq 0$) obviously gives for Y^{-1} the transformation law of a contravariant vector $Y^{-1} \to MY^{-1}M^\dagger$. A contravariant vector Y^ϵ may be constructed as a linear combination of the entries of Y by

$$Y^\epsilon_{ij} = (\hat{R}^\epsilon_h)^{-1}_{ij,kl}Y_{kl} \quad , \tag{6.2}$$

(cf. (3.3)). It may be seen that the contravariant vectors Y^{-1} and Y^ϵ are related by an invariant and central factor,

$$Y^\epsilon = \frac{2+h^2}{2}(det_h Y)Y^{-1} = \frac{2+h^2}{2}\Box_h Y^{-1} \quad . \tag{6.3}$$

A natural definition of the deformed Dirac operator (see [17]) is the following

$$\not{D}_h = \left[\begin{array}{cc} 0 & Y \\ Y^\epsilon & 0 \end{array}\right] \quad . \tag{6.4}$$

This definition is in fact general and is also valid for all the spacetime Q-deformations ($Q{=}q,h$) in [12]; the different cases differ from one another in the explicit expression of Y^ϵ and on the commutation relations among the generators of the $\mathcal{D}_Q$ algebras obtained for the different deformed Y's.

Using eqs. (6.1) and (6.2) we find that the 2×2 matrix Y^ϵ is given by

$$Y^\epsilon = [e_{kl}(\hat{R}^\epsilon_h)^{-1}_{kl,ij}]\partial_{ij} \quad . \tag{6.5}$$

Thus, the Dirac operator can be written as a contraction of the initial generators ∂_{ij} with 4×4 matrices (deformed γ-matrices)

$$\not{D}_h = \gamma_{ij}\partial_{ij} \quad , \quad \gamma_{ij} := \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right) \otimes e_{ij} + \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right) \otimes e_{kl}(\hat{R}^\epsilon_Q)^{-1}_{kl,ij} \, . \tag{6.6}$$

Clearly, the more familiar notation is obtained by replacing the pairs $(ij), (mn)$... by μ, ν...

We have to prove that the h-deformed γ-matrices satisfy the suitable deformed Clifford algebra relation; as in the undeformed case, it must be consistent with the relation $\not{D}_h{}^2 \propto \Box_h$ and with the commutation relation for the derivatives. Hence, the deformed relation for γ-matrices will be written in terms of the metric given for Y, eq. (3.7), and in terms of the 16×16 $\mathcal{R}$-matrix of the exchange algebra for Y, $\mathcal{R}^Y$. This is obtained from the reflection equation for Y (see (2.13))

$$R^{(4)}Y_2R^{(2)\,-1}Y_1 = Y_1R^{(3)\,-1}Y_2R^{(1)} \quad , \tag{6.7}$$

($R^{(3)} = \mathcal{P}R^{(2)}\mathcal{P}$) analogously to (5.5), by identifying the intermediate expression

$$Y_{ia'}Y_{c'd'}R^{(1)}_{b'd',mn}((R^{(2)\,-1})^{t_2})^{-1}_{dc,an}R^{(3)\,-1}_{a'j,b'c'}R^{(4)\,-1}_{ab,ij} = Y_{bc}Y_{dm} \tag{6.8}$$

with the exchange algebra relation $Y_1Y_2\mathcal{R}^Y = Y_2Y_1$. Then,

$$\mathcal{R}^Y_{(ia')(c'd'),(dm)(bc)} = R^{(1)}_{b'd',mn}((R^{(2)\,-1})^{t_2})^{-1}_{dc,an}R^{(3)\,-1}_{a'j,b'c'}R^{(4)\,-1}_{ab,ij} \quad . \tag{6.9}$$

Proposition 6.1

The h-deformed Dirac matrices (6.6) satisfy the deformed Clifford algebra relations

$$\gamma_{ij}\gamma_{mn} + \mathcal{R}^Y_{(ij)(mn),(rs)(kl)}\gamma_{kl}\gamma_{rs} = (2+h^2)g^Y_{h\,mn,ij}I_4 \quad , \tag{6.10}$$

where $\mathcal{R}^Y$ and g_h^Y are given by (6.9) and (3.8) respectively.
Proof :
Using the Weyl-like realization (6.6) for the deformed γ-matrices, eq. (6.10) splits into two equations involving the 2×2 matrices e_{ij}, namely

$$e_{ij}e_{ab}(\hat{R}_h^{\epsilon\,-1})_{ab,mn} + \mathcal{R}^Y_{(ij)(mn),(rs)(kl)}e_{kl}e_{cd}(\hat{R}_h^{\epsilon\,-1})_{cd,rs} = (2+h^2)g^Y_{h\,mn,ij}I_2 \quad , \tag{6.11}$$

$$e_{zt}(\hat{R}_h^{\epsilon\,-1})_{zt,ij}e_{mn} + \mathcal{R}^Y_{(ij)(mn),(rs)(kl)}e_{fg}(\hat{R}_h^{\epsilon\,-1})_{fg,kl}e_{rs} = (2+h^2)g^Y_{h\,mn,ij}I_2 \quad , \tag{6.12}$$

Let us prove eq. (6.11) (eq. (6.12) is proved similarly). Eq. (6.11) may be written in components by using the explicit form of the matrices $(e_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$ so that $(e_{ij}e_{ab})_{\alpha\beta} = (e_{ij})_{\alpha\gamma}(e_{ab})_{\gamma\beta} = \delta_{i\alpha}\delta_{ja}\delta_{b\beta}$. Then, the matrix element $\alpha\beta$ of the *l.h.s.* of the eq. (6.11) is

$$\delta_{i\alpha}(\hat{R}_h^{\epsilon\,-1})_{j\beta,mn} + \mathcal{R}^Y_{(ij)(mn),(rs)(\alpha c)}(\hat{R}_h^{\epsilon\,-1})_{c\beta,rs} \quad . \tag{6.13}$$

Now, using the expression (6.9) for $\mathcal{R}^Y$ and (3.3) for $\hat{R}^\epsilon_h$, one gets for the *l.h.s.*

$$\delta_{i\alpha}\epsilon^t_{h\,\beta\beta'}R^{(3)\,-1}_{j\beta',n'm}\epsilon^t_{h\,n'n} + R^{(1)}_{b'n,sl}((R^{(2)\,-1})^{t_2})^{-1}_{rc,al}R^{(3)\,-1}_{jk,b'm}R^{(4)\,-1}_{a\alpha,ik}\epsilon^t_{h\,\beta\beta'}R^{(3)\,-1}_{c\beta',s'r}\epsilon^t_{h\,s's} \quad . \tag{6.14}$$

contracting pairs of indices in the second term of (6.14)

$$((R^{(2)\,-1})^{t_2})^{-1}_{rc,al}R^{(3)\,-1}_{c\beta',s'r} = ((R^{(2)\,-1})^{t_2})^{-1}_{rc,al}(R^{(2)\,-1})^{t_2}_{\beta's',rc} = \delta_{\beta'a}\delta_{s'l} \quad , \tag{6.15}$$

and using

$$R^{(1)}_{b'n,sl}\epsilon_{h\,sl} = -\epsilon_{h\,nb'} \quad , \tag{6.16}$$

(which follows from (2.4), (2.5)) allows us to write the *l.h.s.* of (6.11) as

$$\epsilon^t_{h\,\beta\beta'}\left(\delta_{i\alpha}\delta_{\beta'k} - R^{(4)\,-1}_{\beta'\alpha,ik}\right)R^{(3)\,-1}_{jk,b'm}\epsilon_{h\,nb'} \quad . \tag{6.17}$$

Since $R^{(4)} = R^{(1)\,\dagger} = R^t_h$, the term in the brackets is $(I - \hat{R}_h^{-1})_{ik,\alpha\beta'}$ which, using the spectral decomposition of $\hat{R}_h$ (eq. (2.4)), is equal to $2P_{h-\,ik,\alpha\beta'}$. Finally, using eq. (2.5) to express P_{h-} in terms of ϵ_h, one gets by direct computation (see eq. (3.8))

$$\delta_{\alpha\beta}\left(-\epsilon_{h\,iz}\epsilon^{-1}_{h\,sz}\right)\left(\epsilon^t_{h\,sk}\hat{R}^{(3)\,-1}_{jk,mb'}\epsilon^t_{h\,b'n}\right) = \delta_{\alpha\beta}D_{h\,is}(\hat{R}_h^{\epsilon\,-1})_{js,mn} = \delta_{\alpha\beta}(2+h^2)g^Y_{h\,mn,ij} \tag{6.18}$$

which coincides with the matrix element $\alpha\beta$ of the *r.h.s.* of eq. (6.11), *q.e.d.*

If instead of the basis $\{e_{ij}\}$ another one is selected, the resulting matrices will be the gamma matrices in another 'spacetime' basis. However, in the deformed case there is not a clear way to define a 'physical' basis. For the q-Minkowski space of [8, 9], there is a central element in the algebra which may be associated with time and the other generators may be grouped by its tensorial properties under the deformed rotation subgroup, so that one may define q-Pauli and q-gamma matrices (see in this respect [8, 37, 38, 17]). However, there is no deformed rotation subgroup for all Lorentz deformations (as *e.g.*, for [31]), and there is no central element that we can

clearly select as time, as in the present case of the h-deformation. Thus, we shall not attempt to introduce any special basis and we shall work with the generators α, β, γ, δ in (2.9). Thus, we shall label the four gamma matrices as γ^α, γ^β, γ^γ, γ^δ and write $\not{D}_h = \gamma^I \partial_I$ $(I = \alpha, \beta, \gamma, \delta)$.

To find their explicit form, we need the expressions of $\hat{R}_h^\epsilon$ and Y^ϵ, which are found from (3.3) and (6.2) for $\mathcal{M}_h^{(1)}$ and $\mathcal{M}_h^{(2)}$. The results are

1. $\mathcal{M}_h^{(1)}$ case:

$$(\hat{R}_h^\epsilon)^{-1} = \begin{bmatrix} h^2+r & -h & -h & 1 \\ h & -1 & 0 & 0 \\ h & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \; ; \; Y^\epsilon = \begin{bmatrix} (h^2+r)\partial_\alpha - h(\partial_\beta + \partial_\gamma) + \partial_\delta & h\partial_\alpha - \partial_\gamma \\ h\partial_\alpha - \partial_\beta & \partial_\alpha \end{bmatrix} \; ; \tag{6.19}$$

$$\begin{gathered} \gamma^\alpha = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ h^2+r & h & 0 & 0 \\ h & 1 & 0 & 0 \end{bmatrix} \quad , \quad \gamma^\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -h & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad , \\ \gamma^\gamma = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -h & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad , \quad \gamma^\delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad . \end{gathered} \tag{6.20}$$

2. $\mathcal{M}_h^{(2)}$ case:

$$(\hat{R}_h^\epsilon)^{-1} = \begin{bmatrix} -h^2 & 0 & 0 & 1 \\ 2h & -1 & 0 & 0 \\ 2h & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad Y^\epsilon = \begin{bmatrix} -h^2\partial_\alpha + \partial_\delta & 2h\partial_\alpha - \partial_\gamma \\ 2h\partial_\alpha - \partial_\beta & \partial_\alpha \end{bmatrix} \quad ; \tag{6.21}$$

$$\begin{gathered} \gamma^\alpha = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -h^2 & 2h & 0 & 0 \\ 2h & 1 & 0 & 0 \end{bmatrix} \quad , \quad \gamma^\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad , \\ \gamma^\gamma = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad , \quad \gamma^\delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad . \end{gathered} \tag{6.22}$$

Let us now discuss the plane wave solutions of the h-K-G operator.

7 h-deformed invariant equations: solutions of the K-G equation

Relativistic invariant equations have been already discussed for the q-deformation [37, 38, 39, 17, 40] and recently the case of the deformed Klein-Gordon and Dirac equations on a deformed homogeneous space (Minkowski) has been considered in [15]. We shall follow here a method similar to that followed in [32]. This means that together with the coordinate and derivative algebras $\mathcal{M}_h^{(j)}$, $\mathcal{D}_h^{(j)}$ (j=1,2) we shall introduce another algebra, the algebra of *momenta* $\mathcal{P}_h^{(j)}$. The generators of $\mathcal{P}_h^{(j)}$ will be the elements of a 2×2 matrix P. We may look at P as the eigenvalues matrix of the derivative matrix (in a naive quantization we may set $P/i\hbar{\sim}Y$ where $\hbar$ is the Planck constant). Thus, P transforms as a 'covariant' vector under an element of $L^{(j)}$, $P \mapsto (M^\dagger)^{-1}PM^{-1}$ (cf. (2.12)).

Since we make the natural assumption $P \propto Y$, it is equally natural to assume that the elements of P neither commute among themselves nor with those of K and Y (cf. [15]). The requirement that these commutation relations are preserved under the Lorentz coaction may be again expressed by reflection equations. As mentioned in Sec. 2, their R-matrix structure in the h-deformed case is unique: their form depends only on the contravariant (eq. (2.10)), covariant (eq. (2.13)) or mixed (eq. (2.14)) character of the objects that they contain. Hence, the commutation relations of the elements in $P^{(j)}$ (generators of $\mathcal{P}_h^{(j)}$) with those of $K^{(j)}$ (id. of $\mathcal{M}_h^{(j)}$) and $Y^{(j)}$ (id. of $\mathcal{D}_h^{(j)}$) are fully determined by the equations

$$R_h^\dagger P_2 R^{(2)\,-1} P_1 = P_1 R^{(3)\,-1} P_2 R_h \quad , \tag{7.1}$$

$$P_2 R_h K_1 R^{(2)} = R^{(3)} K_1 R_h^\dagger P_2 \quad , \tag{7.2}$$

$$R_h^\dagger Y_2 R^{(2)\,-1} P_1 = P_1 R^{(3)\,-1} Y_2 R_h \quad . \tag{7.3}$$

As the commutation relations among the elements of P themselves are the same as those of the derivatives Y, it follows that we can take K and P simultaneously hermitian and that

$$P^2 := \frac{1}{2+h^2} tr_h(P^\epsilon P) \quad , \quad P^\epsilon := (\hat{R}_h^\epsilon)^{-1} P \tag{7.4}$$

is central in the $\mathcal{P}_h^{(j)}$ algebra (compare with the definition of the d'Alembertian, eq. (3.6)) and in $\mathcal{M}_h^{(j)}$, $\mathcal{D}_h^{(j)}$.

The first step in the search for solutions of the deformed equations is to see whether the standard plane wave solutions $e^{-ipx/\hbar}$ are modified for the h-deformed Klein-Gordon equation. To see that they are not, we first introduce the h-Lorentz $L_h^{(j)}$ invariant scalar product for coordinates and momenta

$$(K,P) := tr_h(KP) \quad . \tag{7.5}$$

It is not difficult to check now, using the invariance of the h-trace (3.13) and eqs. (7.1) and (7.2), that (K,P) commutes with the elements of the $\mathcal{M}_h^{(j)}$ and $\mathcal{P}_h^{(j)}$ algebras,

$$[(K,P),K] = 0 = [(K,P),P] \quad . \tag{7.6}$$

Similarly, and using that $tr_{h(1)}(R_{12}\mathcal{P}) = I$ for the matrix R_h, we find

$$Y(K,P) = (K,P)Y + P \quad . \tag{7.7}$$

Iterating this equation we obtain for the n-th power of the scalar product

$$Y(K,P)^n = (K,P)^n Y + n(K,P)^{n-1}P \tag{7.8}$$

so that, on a constant, eq. (7.8) gives the usual rule

$$Y(K,P)^n = n(K,P)^{n-1}P \quad . \tag{7.9}$$

Hence, using the *ordinary* expansion $\exp i(K,P) = \sum_{n=0}^{\infty} \frac{1}{n!}(iK,P)^n$ we check that

$$Y \, \exp i(K,P) = iP \, \exp i(K,P) \quad . \tag{7.10}$$

Taking again the derivative, now using Y^ϵ, we finally obtain

$$\Box_h \, \exp i(K,P) = -P^2 \, \exp i(K,P) \quad , \tag{7.11}$$

where we recall that $\Box_h$ $[P^2]$ is given in (3.6) [(7.4)]. Thus, the deformed plane-wave solutions of the h-K-G equation have the same *form* as in the free case, and a mass-shell $P^2 = m^2c^2$ may be defined similarly. A similar study could be performed for the Dirac equation since the expressions of the gamma matrices are known and the exponential $\exp i(K,P)$ commutes with P, so one can reduce the h-Dirac equation to a matrix one with non-commutative entries P_{ij}, but we shall not do it here.

8 Conclusions

We have studied the h-deformations of the Lorentz group associated with $SL_h(2)$ and the corresponding Minkowski spacetime algebras. The twisted character of these deformations has been shown by exhibiting the twisted nature of $L_h^{(1,2)}$, and it is also measured by the h-dilatation operator s. Our analysis shows that the irreducible representations of deformed Minkowski algebras are quite different for different deformations.

We have also studied the kernels of the h-deformed relativistic equations and given the defining h-Clifford algebra relations and explicit form for the h-Dirac matrices. The analysis of the solutions of the h-K-G equation is facilitated by the fact that the h-Minkowski differential calculi, which are based on the twisted $SL_h(2)$ deformation, have a real structure. We now conclude by comparing our procedure and solutions with those in [15]. There, the deformed momenta are also introduced as the generators of an algebra (of exchange type for a triangular R-matrix). However, and in contrast with our $\mathcal{P}_h^{(j)}$ algebra, these momenta commute with the coordinates and the derivatives. This unnatural simplification of the commutation properties, which would correspond to replacing eqs. (7.2), (7.3) by $P_2K_1 = K_1P_2$ and $Y_2P_1 = P_1Y_2$, produces a more complicated action of the derivatives on the scalar product and (at least within the present scheme) appears to be inconsistent with covariance, since the previous expressions lack the R-matrices necessary to reorder the non-commuting M matrices which transform P, K and Y.

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A Appendix: The 16×16 h-Lorentz R-matrices

The explicit expressions of the 16×16 $\mathcal{R}$-matrices for $L_h^{(1,2)}$ are obtained from eq. (5.5), where $R^{(1)} = R_h$ (eq. (2.1)), $R^{(2)} = \mathcal{P}R^{(3)}\mathcal{P}$, the $R^{(3)}$ are given in (2.8) and $R^{(4)} = R_h^\dagger$ (cf. eq. (2.10)). Thus,

1. $\mathcal{M}_h^{(1)}$ case:

$$\mathcal{R}_h^{(1)} = \left[\begin{array}{cccccccccccccccc}
1 & h & -h & -h^2 & -h & h^2 & h^2+r & h(r-h^2) & h & h^2-r & h^2 & h(h^2-r) & -h^2 & h(h^2-r) & h(r-h^2) & h^4-r^2 \\
0 & 1 & 0 & -h & 0 & h & 0 & r-h^2 & 0 & h & 0 & h^2 & 0 & h^2 & 0 & h(h^2+r) \\
0 & 0 & 1 & h & 0 & 0 & -h & h^2 & 0 & 0 & -h & -h^2-r & 0 & 0 & h^2 & -h(h^2+r) \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & h & 0 & 0 & 0 & -h & 0 & 0 & 0 & -h^2 \\
0 & 0 & 0 & 0 & 1 & -h & -h & h^2 & 0 & 0 & 0 & 0 & h & -h^2-r & h^2 & -h(h^2+r) \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -h & 0 & 0 & 0 & 0 & 0 & h & 0 & h^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -h & 0 & 0 & 0 & 0 & 0 & 0 & -h & h^2-r \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & h & h & h^2 & -h & h^2 & r-h^2 & h(h^2+r) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & h & 0 & h & 0 & h^2+r \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & h & 0 & 0 & -h & h^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -h & h & -h^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \tag{A.1}$$

2. $\mathcal{M}_h^{(2)}$ case:

$\mathcal{R}_h^{(2)} =$

$$
\begin{bmatrix}
1 & 2h & -2h & -4h^2 & -2h & 0 & -h^2 & -2h^3 & 2h & h^2 & 0 & 0 & -4h^2 & 0 & -2h^3 & 3h^4 \\
0 & 1 & 0 & -2h & 0 & 0 & 0 & -h^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2h & 0 & 0 & 0 & 4h^2 & 0 & 0 & 0 & -3h^2 & 0 & 0 & 0 & -6h^3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 2h & 0 & 0 & 0 & -2h & 0 & 0 & 0 & -4h^2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2h & -3h^2 & 4h^2 & -6h^3 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3h^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2h & 0 & -h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3h^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2h & 2h & -4h^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\tag{A.2}
$$

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